Foundations of Applied Math
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A Basis for Simulation and Analysis of Differential Pursuit Games

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ABSTRACT: The theory of differential games is explored in relation to two-player pursuit games. An examination of models based on games of degree is compared with models based on games of kind with consideration to maneuvering strategies. An unconstrained two-dimensional space is used as the playing surface. The model for a simple single, continuous turn is considered along with the formation of a strategy based on a series of simple continuous turns. Energy considerations are used for terminating conditions. An exploration of state and control variables for games of degree and games of kind is performed, with the basis for a game of degree showing greater promise for expansion of this model to *n* dimensions with applied probability modeling.

1. Introduction: Differential games (DG) theory was born in the 1960's primarily for applications regarding combat situations [1]. Isaacs pioneered this field of research as an alternate means by which combat scenarios could be formulated. Indeed, much of DG theory evolves directly from the older discrete games theory (GT). Typically the process of DG modeling involves a "smoothing" of the game to fit continuous models rather than discrete ones. Similarly, many differential games can be modeled by GT. It is apparent that the sequences of some types of games (i.e. American football) could be sufficiently modeled by either.

The concept of statistical decisions is an extension of decision theory, the latter of which came into prominence in the 1920's [2]. Given statistical measures of the next turn in a game, decisions regarding an optimizing strategy can be met. This paper does not provide much more than a qualitative analysis on the mathematics of statistical decision theory, but it easily noticeable that the fundamental principles for statistical decision theory apply conveniently to a stochastic simulation for DG theory.

A classic DG is considered here, and it was largely formed from a modification to the classic "homicidal chauffeur" game posed in [1]. The initial motivation for the project was to model aerial combat between dissimilar aircraft in three-dimensional space. However, the game for combat between aircraft is rather messy, as pointed out by Aboufadel in [3]. Therefore, the decision to limit the scope to two dimensions was made, with the underlying motivation then transforming into a model for pursuit of a cheetah and an antelope. Lacking the zoological knowledge of cheetah hunting methods, and given the fact that not many antelopes can perform calculus while running for their lives, a more realistic example of a car chase on an open parking lot may be considered if the reader finds the former example too far a stretch of the imagination.

2. One-Player Differential Games: A one-player DG serves as a good approximation for a race car driver in an autocross. An autocross is a form of race where a driver is timed as he or she races through a complex course involving sharp turns, slaloms, and straight-aways. Therefore, this turns into a one-player DG of degree, where the payoff is considered to be the time to completion of the course, *t*. Let us be an observer to this game with a wager on the time to completion for each certain driver. We can therefore write a general DG in terms of state and control variables as follows:

$$x = (x_1, x_2, x_3, x_4)$$
$$\varphi = (\varphi_1, \varphi_2)$$

Where x_1 and x_2 are the x- and y-coordinates of the vehicle, respectively. It follows that:

$$\dot{x}_3 = A\varphi_1$$
$$\dot{x}_4 = W\varphi_2$$

Here, *A* represents the maximum possible acceleration of the vehicle, *W* represents the maximum rate of change of the angle of turn, and φ_1 and φ_2 represent control variables for *A* and *W*, respectively. Thus, it is apparent that:

$$0 \le \varphi_1 \le 1$$
$$-1 \le \varphi_2 \le 1$$

This gives a complete description of what the car is doing as far as the driver is concerned, but as observers it might be important to add another state variable, x_5, to the game to denote the direction the vehicle is facing. We can then write the game in terms of kinematic equations (KE):

$$\dot{x}_1 = x_3 \cos x_5$$
$$\dot{x}_2 = x_3 \sin x_5$$
$$\dot{x}_3 = A\varphi_1, \quad 0 \le \varphi_1 \le 1$$
$$\dot{x}_4 = W\varphi_2, \quad -1 \le \varphi_2 \le 1$$
$$\dot{x}_5 = x_4 x_3$$

The full derivation of this, as with additional assumptions not considered here, is detailed in p.6 of Isaacs [1]. However, one critical assumption that will be thrown out in the above derivation is that this vehicle has no transmission. Indeed, one who drives a vehicle with a manual transmission knows that there are more ways to accelerate than simply hitting the accelerator. For the purposes of a qualitative analysis of applied statistical decision theory, we will examine what would occur should this not be the case.

It might be apparent that the driver of this car (which we assume to have some sort of transmission system) can accelerate the vehicle in any of four available gears. We can then say:

$$S = \{A_1, A_2, A_3, A_4\},$$
$$A \in S$$

where A_X denotes the maximum possible acceleration in gear *X*. Since:

$$P(A \in S) = 1,$$

we can write that:

$$P(A = A_1) = p_1$$
$$P(A = A_2) = p_2$$
$$P(A = A_3) = p_3$$
$$P(A = A_4) = p_4$$

If we look further, we could say that the driver will not attempt to accelerate the vehicle in 3rd or 4th gear since higher gears accelerate more slowly, and consequently $p_3=p_4=0$. Furthermore, say we can reduce the range of the driver's control variable such that:

$$0.8 \le \varphi_1 \le 1$$

Then we can map two subspaces G_1 and G_2 in the playing space G where the driver is likely to be after a given time t.

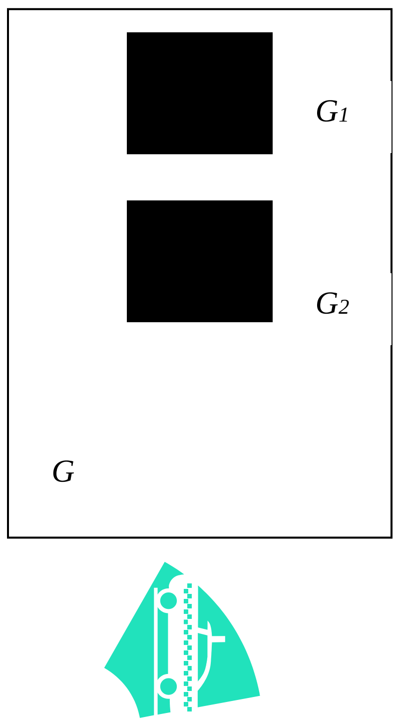

Figure 1

Thus, if we knew this sort of information ahead of time, we could place a wager on the car being in G_1 or G_2 with probability p_1 or p_2, respectively. This is an application of the underlying theory involving statistical decision theory, and this qualitative analysis will be as far as we go with applying probability to the DG models in this paper. Further applications of statistical decision theory in relation to GT is discussed in depth in the Blackford text [2].

3. Time Distances: In terms of the KE's of the DG at hand, it is very inconvenient to write the state variables in terms of long, complex equations for the motion of a player. Thus, it is useful to decompose the physical problem into terms of a "time distance," or rather the amount of time it would take for a player to get to a given point on the playing surface given certain speed and acceleration characteristics. Then if the game were to arrive at a certain point *P* on a playing surface, the payoff would be the total time *T* and the terminating condition would be when the time distance $\Delta t=0$. For the two-player pursuit game, that arbitrary point becomes the evader *E* which may or may not move, and the player attempting to get to *E* is the pursuer, *P*. We will no longer use *P* to denote probability, as our discussion involving that aspect of GT is complete. We will later discuss the use of these time distances to denote the total difference between *E* and *P* at a given time in the game as state variables, and the time distances will be broken into components corresponding to the x- and y-coordinates. We will also see how this decomposition becomes useful in determining a turn strategy.

4. Energy Relations, Player Conditions, and Terminating Conditions: In order to properly model the two-player game, as we did with the one-player game, it is important to consider the conditions by which the players' motion is governed over a period of time and the terminating conditions of the game. In this game, we will consider two terminating conditions: one, where *P* captures *E*, and two, where *P* does not have enough energy to close the time distance to *E*. Therefore, we need to give some though to the energy relations that we have. Consider:

$$\begin{aligned} F &= mA\varphi_1 \\ D &= F \end{aligned},$$

at maximum speed. Thus, to run at maximum speed to counter friction and drag (denoted by *D*), the animal (or car) must apply full acceleration. It then becomes sufficient to write the energy relation in terms of this acceleration term:

$$\begin{aligned} E &= Fd \\ F &= mA\varphi_1 \\ E &= mA\varphi_1 d \end{aligned},$$

where *d* is the displacement. Continuing:

$$\begin{aligned} \frac{\Delta E}{\Delta t} &= m(A\varphi_1)\frac{\Delta d}{\Delta t} \\ d &= d(t) \end{aligned}$$

Replacing *d* by *d** to remove a potential confusion, we take the limit as Δt goes to zero.

$$\lim_{\Delta t \to 0}\left(\frac{\Delta E}{\Delta t} = m(A\varphi_1)\frac{\Delta d^*}{\Delta t}\right)$$

$$= \frac{dE}{dt} = m(A\varphi_1)\frac{dd^*}{dt} \qquad (1)$$

From basic physics we know:

$$\begin{aligned} \frac{dd^*}{dt} &= v(t) = \int_t (A\varphi_1)dt \\ &= (A\varphi_1)t + v_0 \end{aligned}$$

Plugging this term into (1) we can see that:

$$\frac{dE(t)}{dt} = m\left[(A\varphi_1)^2 t + (A\varphi_1)v_0\right] \qquad (2)$$

This may also be confirmed dimensionally. We will use (2) as the governing equation for the terminating condition. We will also make a few assumptions about the players themselves. Assume that *P* is faster than *E*, that is to say:

$$A_P > A_E$$

Then we can see that it uses up its energy faster. Also assume that *P* and *E* have some finite initial energies. To avoid confusion, we will refer to energy as *M* from here forth. Therefore, *P* and *E* have initial energies M_{Po} and M_{Eo}. There is no definite inequality between these. It may be that $M_{Po}>M_{Eo}$, but for the purposes of this game we will state that if such is the case, then $M_{Po}>>M_{Eo}$ cannot be the case, otherwise it is almost certain that *P* will capture *E* simply by wearing it into the ground. (An interesting aside in relation to the military nature of the Isaacs text, if the latter were the case and we took *M* to be the munitions reserve of an army, this would be the case of a war of attrition where one army simply has more ammo (or men) to expend on its opponent.) The assumption will also be made that the energy remaining has no impact on the speeds at which the players can move, similar to a car with a fuel tank.

5. One-Dimensional, Two-Player Differential Games of Pursuit: Admittedly there is no real challenge or interest in a two-player DG of pursuit that follows only one dimension, but it forms a necessary basis for more complex games. Consider *E* and *P* to be on a one-dimensional playing surface with a given initial time distance δt:

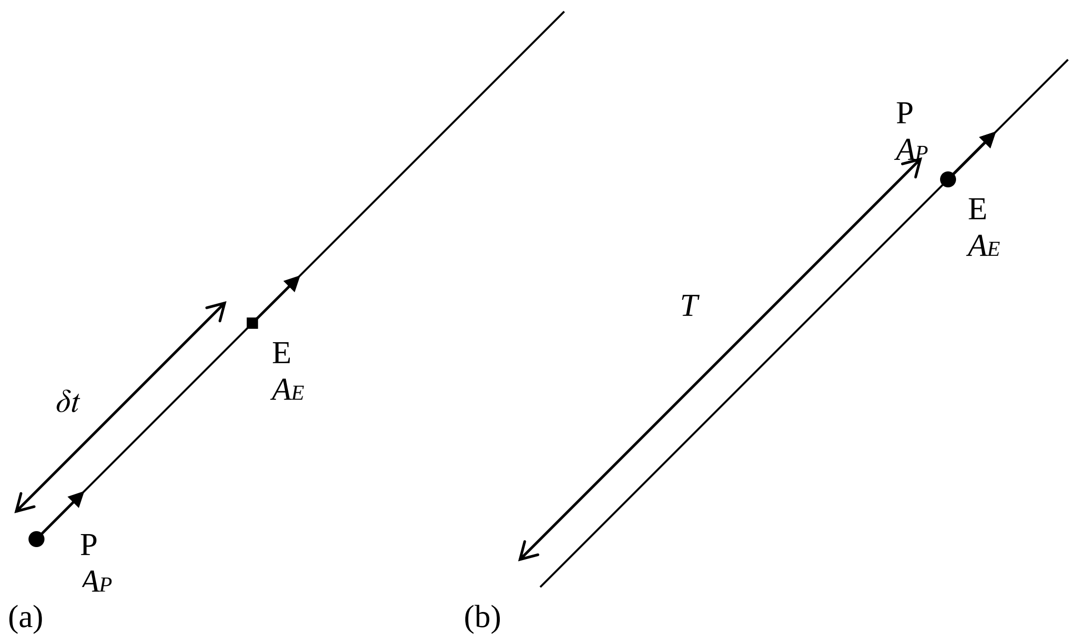

Figure 2

Given the situation in Figure 2a we can determine whether the situation in Figure 2b is possible. A transformation to a one-player game is performed:

$$(A\varphi_1) = A_P - A_E$$

Plugging this, and $t=T$ into (2) will yield a depletion in energy for *P*, which can then be subtracted from M_{Po}. If the resultant value is negative, we then know that *P* cannot capture

E. The matter of finding *T* is trivial; simply use basic physical laws to solve for the distance traveled by each *P* and *E* and calculate the intercept point.

However, if plugging the time *T* into a transformed equation for *P* does not yield a capture for *E*, the game is not necessarily over. We must check if *E* also had enough energy to run that given distance. If so, then the game has been won in *E's* favor. If not, then it is necessary to perform some tedious calculations to see where *E* runs out of energy. If this is greater than the time at which *P* runs out of energy, then the game is also won in *E's* favor. If not, then the scenario arises where *E* has run out of energy and is "dead in the water" and can merely sit and wait for *P* to capture and win the game in *P's* favor.

Assume that this last scenario is in fact possible. We may wish to optimize *E's* motion to use as little energy as possible. Thus, for an initial time t_1 *E* does not run at $\varphi_1=1$ as in the previous example.

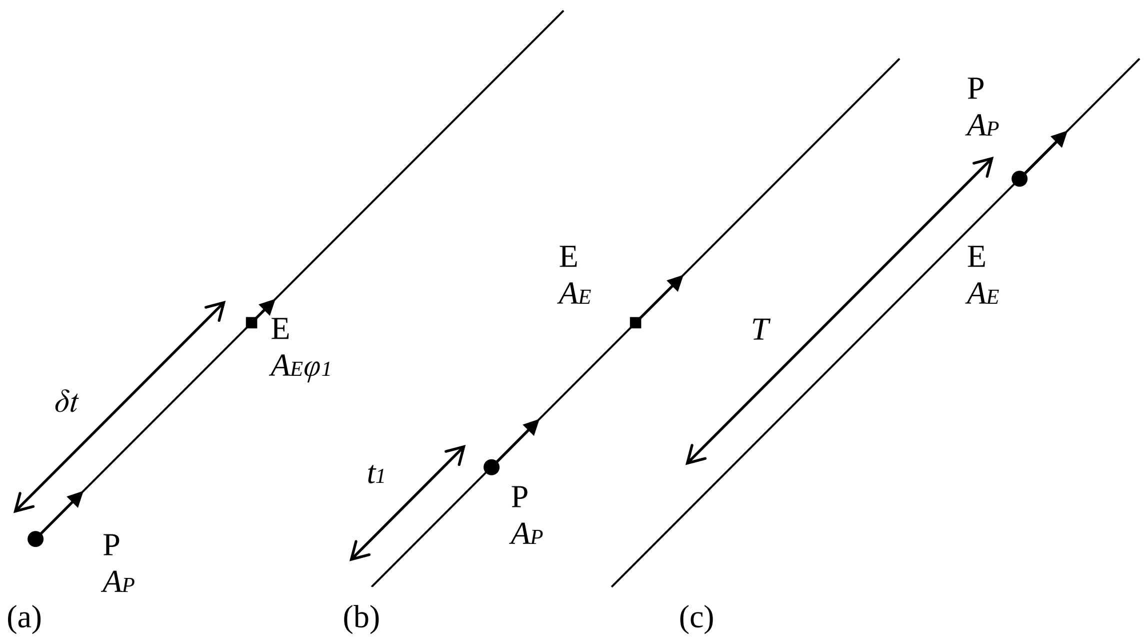

Figure 3

In Figure 3a the situation is the same as in Figure 2a, but *E* is not running at full speed. Instead it is conserving energy until *P* has gotten closer and lost some of its energy. Then, when we arrive at Figure 3b, *E* runs at full speed. Thus, we can consider the stage from (b) to (c) as the same scenario as Figure 2, and we can use the same methods. However, it is important that we subtract the energy spent from going from (a) to (b) in Figure 3 from the initial energies that *E* and *P* had, namely M_{Po} and M_{Eo}.

While this situation serves little practical purpose, the concepts of discretizing the DG into stages, and the subsequent superposition of these decomposed stages into a full DG is fundamental to the turning concepts that will be discussed in the next section. However, before turning to a full two-dimensional model, a few comments on the decisions made in the second portion of this one-dimensional system are worth making. Here we see the choice of a minimizing strategy by *E*. It is common in GT for one player to attempt to minimize the chances of the other player to win. This other player consequently attempts to maximize its chances of winning. This is known as a minimax problem [2]. In a normalized discrete game, it is not apparent who is minimizing and who is maximizing from the onset. Instead, it changes each turn depending on the conditions of the game (one who has played the popular game *Risk* understands the dynamic nature of minimizing and maximizing strategies for each player). However, in DG's of pursuit it stands constant that *E* is minimizing and *P* is maximizing [1]. This is (somewhat confusingly) converse to the actual goals in terms of our DG, where *E* minimizes by maximizing the time distance and *P* does the exact opposite. In DG theory it is not always clear due to the continuous nature of the game who performs the driving strategy. However, in our unconstrained pursuit model, *P* has no way to set a trap for *E* as if there were some forms of constrictions on the playing surface. Thus, we will hereby state that *E* determines the driving strategy which *P* must follow.

6. Two-Player Pursuit Games with Turning: It is more interesting to consider the scenario regarding a turn by *E* with *P* in pursuit. It stands to reason, barring any physical considerations such as friction, that the faster animal has a larger turning radius. We will offer the problem first graphically then mathematically. Consider the case in Figure 2 or Figure 3 (we have already shown that by superposition the latter can be transformed into the former). Now superpose a turn by *E* on the playing field. We will neglect any reaction by *P* to this turn.

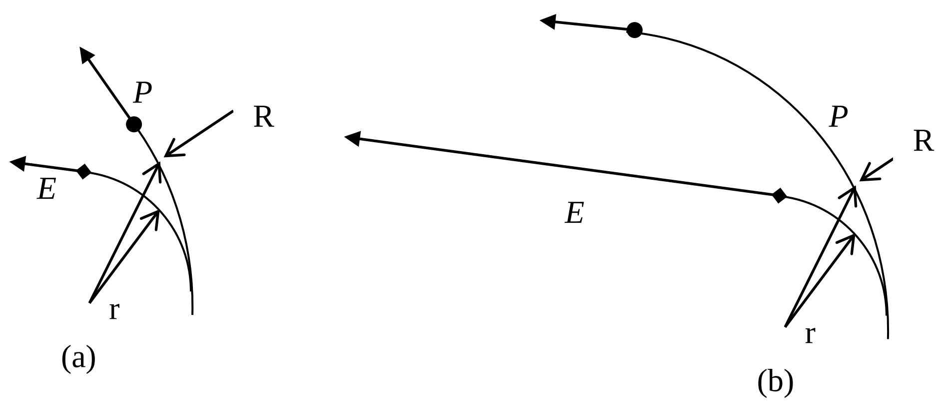

Figure 4

In Figure 4 we see a simple continuous turn. In Figure 4a, *E* has turned by a specified angle θ_1 and *P* has turned by θ_2 (angles not shown for clarity.) We will show that $\theta_1 > \theta_2$:

$$
\begin{aligned}
& r\theta_1 = kR\theta_2 \\
& \frac{r\theta_1}{R\theta_2} = k \\
& k = {x_3^P}\big/{x_3^E} \\
& x_3^P > x_3^P, so \\
& \theta_1 > \theta_2
\end{aligned}
$$

So at the instant that *E* has gone the desired angle θ_1, *P* has only gone an angle of θ_2. If *E* continues straight at this point, then *P* will lose some ground on *E* by necessity of completing the full angle θ_1.

At Figure 4b, *P* has gone the required angle to be parallel with *E*. In the figure below (Figure 5) a very interesting scenario is at hand.

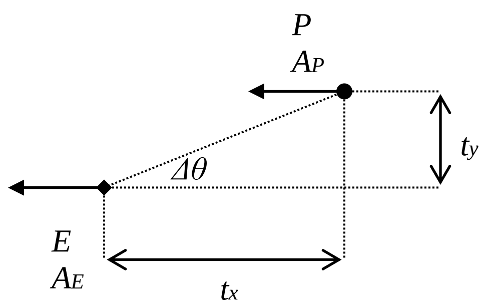

Figure 5

Given the above considerations of decomposing the game into discrete steps and recombining them to form a full DG, we know that at this phase we can almost consider this the same scenario as in section 5. However, there is a critical difference: the distance t_y. If we define a manifold in the DG sense to mean an *n-1*-dimensional surface in the Euclidean *n*-space that is the playing field, we can see that *E* and *P* are no longer on the same manifold. This means that *P* will have to adjust to get back onto the same manifold as *E*. To do so, *P* must perform a turn that is exactly $\Delta\theta$ greater than θ_1. This means that if *E* were to turn again to its left, then it would lose exactly $\Delta\theta$ on the turning angle to *P* but gain t_y in time distance. This scenario may lead to a terminal scenario in favor of *P*. On the other hand, however, if *E* turns to the right then it gains $\Delta\theta$ in angle, but loses t_y in time distance. This is where the concept of a turning strategy enters the game.

There is no definite relationship to define which is the better turn. Clearly if *E* makes a second turn with an angle less that $\Delta\theta$ in either direction then it has lost time distance on *P*. Instead, the relationship between the turning radii will be the determining factor for the strategy. If it is such that larger angles of turn cause the difference t_y to grow every time, then an alternating turn strategy (left-right-left-right) may be the best choice. If however, the loss of the angle is negligible, then clearly a continual turn strategy is optimal. We will continue with a more mathematical approach to the problem:

$$(r,R) \propto (W\varphi_2)^{\alpha} (x_3)^{\beta}$$

It turns out through dimensional analysis that $\alpha=-1$ and $\beta=1$. Thus,

$$(r,R) = \frac{1}{W\varphi_2}(A\varphi_1 t)$$

For a given turn through an angle θ_1, we have that in relation to the beginning of the turn, for E and P respectively:

$$\Delta y = r\sin\theta_1$$
$$\Delta y = R\sin\theta_1,$$

and the difference between these two y-offsets is:

$$t_y = \Delta(\Delta y) = (R-r)\sin\theta_1, \qquad (3)$$

so that as R-r increases, so does the distance t_y. Through a similar process, one can obtain a similar term for the x-offset:

$$\Delta x = r - r\cos\theta_1$$
$$\Delta x = R - R\cos\theta_1$$
$$t_x = \Delta(\Delta x) = (R-r)(1-\cos\theta_1)$$

So as θ_1 is small, t_x is small as well.

If we plug the relationships for R and r into (3) we can conclude that if the speed of P is much larger than the speed of E, then correspondingly their difference between R and r is large as long as W_E and W_P are close. Similarly, if W_P is small (corresponding to a slow rate of turn) then R is again much larger than r. However, we can see a better relationship to determine a strategy for turning. The *sin* term is bounded on the closed interval [-1,1]. Therefore, the driving term in the system is R-r. Thus, regardless of the relationship of turn radii, E cannot gain a significant time distance by alternating turns.

Before continuing on to write the DG completely in terms of its KE's, it is worth reinforcing the superposition principle regarding turn strategies. At the end of each turn, the game essentially can be started over with the new initial conditions being the new coordinates after the turn, and the new total energy left after the previous maneuver. In this manner a full DG can be written as several smaller DG's.

7. Full Kinematic Equations for the Pursuit Game with Simple Turns: It suffices now to write the DG in terms of its KE's. Before blindly stating them, however, we have discussed nothing of the differences between modeling this game as a game of kind or as a game of degree. In a game of kind there are a finite number of outcomes. In this example, the payoff is either capture or evasion. In a game of degree, the payoff is continuous [2]. Here, this would represent the amount of time by which E evades, with zero representing capture. Since we already need to keep track of key components such as energy and time distance, we may consider these to be state variables. Thus, define x_6 to be energy, and x_7 and x_8 to be the x- and y-components of time distance. We therefore add the following to the state variables as discussed in section 2:

$$\dot{x}_6 = m\left[(\dot{x}_3)^2 t + (\dot{x}_3)v_0\right]$$
$$x_7 = t_x = (R-r)(1-\cos\theta_1)$$
$$x_8 = t_y = (R-r)\sin\theta_1$$

Upon expanding x_7 and x_8 with the functions derived earlier for R and r, and introducing the term $\theta_1 = \theta\varphi_3$ since θ_1 is of *E's* choosing, we have:

$$x_7 = \left(\frac{1}{W_P\varphi_2^P}x_3^P - \frac{1}{W_E\varphi_2^E}x_3^E\right)\left(1-\cos\left(\theta\varphi_3^E\right)\right)$$
$$x_8 = \left(\frac{1}{W_P\varphi_2^P}x_3^P - \frac{1}{W_E\varphi_2^E}x_3^E\right)\left(\sin\left(\theta\varphi_3^E\right)\right)$$

But suppose we go one step further and say that the control variables are some unknown functions of time. We can then write the complete KE's for the entire game as follows:

$$\dot{x}_1 = x_3\cos x_5$$
$$\dot{x}_2 = x_3\sin x_5$$
$$\dot{x}_3^{E,P} = A_{E,P}\varphi_1^{E,P}, \quad 0 \le \varphi_1^{E,P} \le 1$$
$$\dot{x}_4^{E,P} = W_{E,P}\varphi_2^{E,P}, \quad -1 \le \varphi_2^{E,P} \le 1$$
$$\dot{x}_5 = x_4 x_3$$
$$\dot{x}_6 = -m\left[(\dot{x}_3)^2 t + (\dot{x}_3)v_0\right]$$

$$\dot{x}_7 = \left(\begin{array}{l} -\dfrac{A_P\varphi_1^P(t)t\left(\dfrac{d\varphi_2^P(t)}{dt}\right)}{W_P\left(\varphi_2^P(t)\right)^2} + \dfrac{A_P\left(\dfrac{d\varphi_1^P(t)}{dt}\right)t}{W_P\left(\varphi_2^P(t)\right)} + \dfrac{A_P\varphi_1^P(t)}{W_P\varphi_2^P(t)} + \\ \dfrac{A_E\varphi_1^E(t)t\left(\dfrac{d\varphi_2^E(t)}{dt}\right)}{W_E\left(\varphi_2^E(t)\right)^2} - \dfrac{A_E\left(\dfrac{d\varphi_1^E(t)}{dt}\right)t}{W_E\left(\varphi_2^E(t)\right)} - \dfrac{A_E\varphi_1^E(t)}{W_E\varphi_2^E(t)} \end{array}\right)\times$$
$$\left(1-\cos\left(\theta\varphi_3^E(t)\right)\right) + \left(\frac{A_P\varphi_1^P(t)t}{W_P\varphi_2^P(t)} - \frac{A_E\varphi_1^E(t)t}{W_E\varphi_2^E(t)}\right)\sin\left(\theta\varphi_3^E(t)\right)\theta\left(\frac{d\varphi_3^E}{dt}\right)$$

$$\dot{x}_8 = \left(\begin{array}{l} -\dfrac{A_P\varphi_1^P(t)t\left(\dfrac{d\varphi_2^P(t)}{dt}\right)}{W_P\left(\varphi_2^P(t)\right)^2} + \dfrac{A_P\left(\dfrac{d\varphi_1^P(t)}{dt}\right)t}{W_P\left(\varphi_2^P(t)\right)} + \dfrac{A_P\varphi_1^P(t)}{W_P\varphi_2^P(t)} + \\ \dfrac{A_E\varphi_1^E(t)t\left(\dfrac{d\varphi_2^E(t)}{dt}\right)}{W_E\left(\varphi_2^E(t)\right)^2} - \dfrac{A_E\left(\dfrac{d\varphi_1^E(t)}{dt}\right)t}{W_E\left(\varphi_2^E(t)\right)} - \dfrac{A_E\varphi_1^E(t)}{W_E\varphi_2^E(t)} \end{array}\right)\times$$
$$\left(\sin\left(\theta\varphi_3^E(t)\right)\right) + \left(\frac{A_P\varphi_1^P(t)t}{W_P\varphi_2^P(t)} - \frac{A_E\varphi_1^E(t)t}{W_E\varphi_2^E(t)}\right)\cos\left(\theta\varphi_3^E(t)\right)\theta\left(\frac{d\varphi_3^E}{dt}\right)$$

These equations then represent the DG for pursuit for continuous turns. As previously mentioned, since we can decompose a game into smaller games, we can use these equations for a game with any amount of turns of any radii. If a more complex turn is to be considered, say with a non-constant radius, then two options exist: define φ_2 as changing with t, or decompose the game into two smaller games at the point where the radii change. If the latter method is chosen, it is important to retain smoothness between the curves. However, these equations will only work for circular turns; other types of turns will require slight modifications.

The basis is now laid for an extension to three dimensions. From the appearance of the last two KE's, it is clear that many avoid this procedure and instead model three-dimensional processes with two-dimensional ones. Notwithstanding, symbolic manipulation programs nowadays have sufficient capability to perform some of the tedious operations automatically, and an extension of this project to three-dimensions is a possible avenue of future research. Alternatively, this model could be extended while remaining in two dimensions by adding constraints to the playing surface. This model also lacked some physical constraints such as a maximum speed and friction between the playing surface and the player, which could easily be added to this system. Furthermore, this model could be used as an exercise in statistical decision theory, and the probability subspaces could be mapped onto the playing space as was done in section 2. Lastly, this is a setup for a stochastic simulation of complex physical processes that may be modeled using differential games. While there is ample discussion on GT and statistical decision theory, there is relatively little research on DG as a separate branch of GT, and even less on statistical decision theory as applied to DG.

If one were to apply statistical decision theory to a DG of this sort on a constrained playing field, some assumptions that are valid in this unconstrained field will not hold true. While it is true that a DG can be decomposed into discrete steps, optimizing a minimax problem cannot. If one were to introduce a wall in the playing surface, then it stands to reason that E may be beneficial to make a turn in one direction at one point, but this may only be a setup for a trap by P. Thus, the optimization of a strategy is a topic not considered in this paper, but discussed in some detail in [1] and [2], the latter as applicable to GT. However, such problems were not the objective of this paper; instead, merely providing a basis upon which to start was the intent. To this extent I feel that it has succeeded.

8. Acknowledgements: I would like to thank Dr. Peter R. Kramer for his advice and guidance on this project, which spawned as an alternative assignment for a high-level undergraduate applied math class. I would also like to thank Dr. Edward Aboufadel of Grand Valley State University, MI, for his recommendations for avenues upon which to look for this project. This project was unfunded, save for the time spent by myself for writing, and the gracious time spent by Dr. Kramer for grading and advising.

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